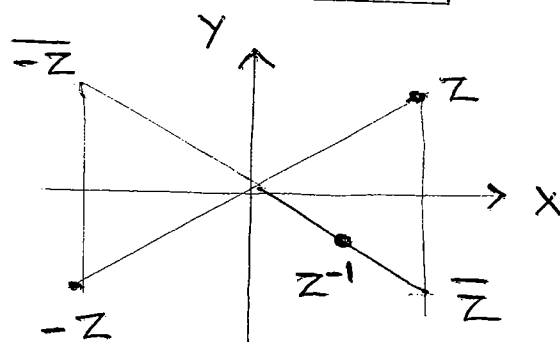


CALCULUS II TUTORIAL (1) : SOLUTIONS

① $z = x + iy \Rightarrow \bar{z} = x - iy,$

$-z = -x - iy, \quad \overline{-z} = -x + iy$

$\frac{1}{z} = \frac{1}{x+iy} \frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2} = \frac{\bar{z}}{|z|^2}$



②

$|z-i| = 2|z-1| \Rightarrow |(x+iy)-i| = 2|(x+iy)-1|$

$\Rightarrow |x+i(y-1)|^2 = 4|(x-1)+iy|^2$ (squaring)

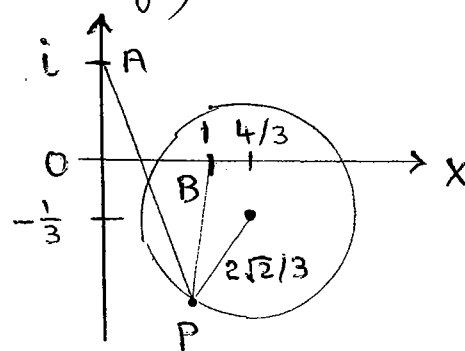
$\Rightarrow x^2 + (y-1)^2 = 4\{(x-1)^2 + y^2\}$

$\Rightarrow x^2 + y^2 - 2y + 1 = 4(x^2 - 2x + 1 + y^2)$

$\Rightarrow 3x^2 + 3y^2 - 8x + 2y + 3 = 0$

$\Rightarrow x^2 - \frac{8x}{3} + \frac{16}{9} + y^2 + \frac{2}{3}y + \frac{1}{9} = \frac{16}{9} + \frac{1}{9} - 1$

$\Rightarrow (x - \frac{4}{3})^2 + (y + \frac{1}{3})^2 = \frac{17}{9} - 1 = \frac{8}{9} = (\frac{2\sqrt{2}}{3})^2$



This is circle with centre $(\frac{4}{3}, -\frac{1}{3})$ and radius $\frac{2\sqrt{2}}{3} \approx 0.9$

Geometrically : The locus of P such that $|PA| = \lambda |PB|$ for any constant λ and fixed points A and B is circle.

Proof : put A at $(0,0)$, B at $(0,b) \Rightarrow x^2 + y^2 = \lambda^2 \{(b-x)^2 + y^2\}$

$\Rightarrow \left\{x - \frac{b\lambda^2}{\lambda^2-1}\right\}^2 + y^2 = \frac{b^2\lambda^2}{(\lambda^2-1)^2}$

③

$\frac{1}{x+iy} = 1 - \frac{1}{a+ib} = \frac{a+ib-1}{a+ib} \Rightarrow x+iy = \frac{a+ib}{(a-1)+ib}$

Multiplying by conjugate of denominator :

$x+iy = \frac{a+ib}{(a-1)+ib} \times \frac{(a-1)-ib}{(a-1)-ib} = \frac{a(a-1)-i^2b^2 + i[b(a-1)-ab]}{(a-1)^2+b^2}$

Equate real and imaginary parts $\Rightarrow \frac{a(a-1)+b^2-i b}{(a-1)^2+b^2}$

$x = \frac{a(a-1)+b^2}{(a-1)^2+b^2}, \quad y = -\frac{b}{(a-1)^2+b^2}$

$$\textcircled{4} \frac{(2+i)^3}{(3+4i)^3} = \frac{(2+i)(4+4i+i^2)}{(3+4i)(9+24i+16i^2)} = \frac{(2+i)(3+4i)}{(3+4i)(-7+24i)} = \frac{2+i}{-7+24i}$$

We now multiply by the conjugate of the denominator:

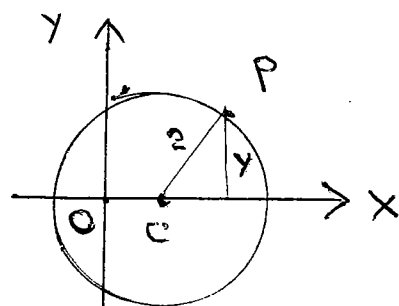
$$\frac{2+i}{-7+24i} \times \frac{-7-24i}{-7-24i} = \frac{-14-55i-24i^2}{(-7)^2+(24)^2} = \frac{10-55i}{49+576} = \frac{2-11i}{125}$$

$$\textcircled{5} \text{ (a) } |z-1| = 2 \Rightarrow |(x-1)+iy| = 2$$

$$\Rightarrow (x-1)^2 + y^2 = 4$$

i.e. circle with centre (1, 0), radius 2

Geometrically: $|z-1|$ is distance PC, so result is obvious.

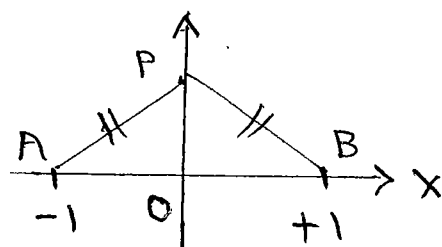


$$\text{ (b) } |z+1| = |z-1| \Rightarrow |(x+1)+iy| = |(x-1)+iy|$$

$$\Rightarrow (x+1)^2 + y^2 = (x-1)^2 + y^2$$

$$\Rightarrow x^2 + 2x + 1 + y^2 = x^2 - 2x + 1 + y^2$$

$$\Rightarrow x = 0 \text{ (i.e. } y\text{-axis)}$$

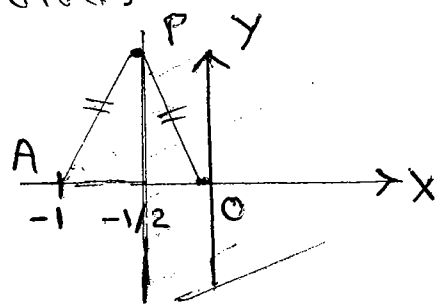


Geometrically: $|z-1|$ is distance PB and $|z+1|$ is distance PA, so this result is obvious

$$\text{ (c) } |z+1| \geq |z| \Rightarrow |(x+1)+iy| \geq |x+iy|$$

$$\Rightarrow (x+1)^2 + y^2 \geq x^2 + y^2 \Rightarrow 2x + 1 \geq 0$$

$$\Rightarrow \text{must be in shaded region } x \geq -\frac{1}{2}$$



Geometrically point P on boundary of region has $PA = PO$ since $|z+1| = |z-(-1)|$ is distance PA etc.